The Mean and Variance Of A Random Variate Plus/Times A Constant

Gary Schurman, MBE, CFA

September, 2010

The Mean and Variance of the Random Variate

Let's begin by defining X as a random variate, N as the number of possible values and $P(X = X_i)$ as the probability that the random variate X will assume the observed value X_i. The probabilities sum to one such that...

$$\sum_{i=1}^{N} P(X = X_i) = 1 \tag{1}$$

The first moment of the distribution of the random variate X is the expected value of X, which is the probability-weighted sum of all possible values of the random variate. The equation for the expected value of the random variate is...

$$\mathbb{E}\Bigg[X\Bigg] = \sum_{i=1}^{N} X_i P(X = X_i) \tag{2}$$

The second moment of the distribution of the random variate X is the expected value of X^2, which is the probability-weighted sum of the square of all possible values of the random variate. The equation for the expected value of the random variate squared is...

$$\mathbb{E}\Bigg[X^2\Bigg] = \sum_{i=1}^{N} X_i^2 P(X = X_i) \tag{3}$$

The mean of X is the first moment of the distribution. The equation for the mean is...

$$mean = \mathbb{E}\Bigg[X\Bigg] \tag{4}$$

The variance of X is the second moment of the distribution minus the square of the first moment. The equation for variance is...

$$variance = \mathbb{E}\Bigg[X^2\Bigg] - \Bigg[\mathbb{E}\Bigg[X\Bigg]\Bigg]^2 \tag{5}$$

The Random Variate Plus a Constant

We want to determine the mean and variance of the random variate plus a constant. Note that if the constant is to be subtracted just change the constant's sign from positive to negative. The equation for the first moment of the distribution is...

$$\begin{aligned}\mathbb{E}\Bigg[X + \alpha\Bigg] &= \mathbb{E}\Bigg[X\Bigg] + \mathbb{E}\Bigg[\alpha\Bigg] \\ &= \mathbb{E}\Bigg[X\Bigg] + \alpha \end{aligned} \tag{6}$$

The equation for the second moment of the distribution is...

$$\begin{aligned}\mathbb{E}\Big[(X+\alpha)^2\Big] &= \mathbb{E}\Big[X^2+2X\alpha+\alpha^2\Big] \\ &= \mathbb{E}\Big[X^2\Big]+\mathbb{E}\Big[2X\alpha\Big]+\mathbb{E}\Big[\alpha^2\Big] \\ &= \mathbb{E}\Big[X^2\Big]+2\,\alpha\,\mathbb{E}\Big[X\Big]+\alpha^2 \end{aligned} \tag{7}$$

The equation for the distribution mean is...

$$\begin{aligned} mean &= \mathbb{E}\Big[X+\alpha\Big] \\ &= \mathbb{E}\Big[X\Big]+\alpha \end{aligned} \tag{8}$$

The equation for the distribution variance is...

$$\begin{aligned} variance &= \mathbb{E}\Big[(X+\alpha)^2\Big]-\Big[\mathbb{E}\Big[X+\alpha\Big]\Big]^2 \\ &= \mathbb{E}\Big[X^2\Big]+2\,\alpha\,\mathbb{E}\Big[X\Big]+\alpha^2-\Big[\mathbb{E}\Big[X\Big]+\alpha\Big]^2 \\ &= \mathbb{E}\Big[X^2\Big]+2\,\alpha\,\mathbb{E}\Big[X\Big]+\alpha^2-\Big[\mathbb{E}\Big[X\Big]\Big]^2-2\,\alpha\,\mathbb{E}\Big[X\Big]-\alpha^2 \\ &= \mathbb{E}\Big[X^2\Big]-\Big[\mathbb{E}\Big[X\Big]\Big]^2 \end{aligned} \tag{9}$$

Note that the mean of the random variate plus a constant is the mean of the random variate (equation (4)) plus the constant and the variance of the random variate plus a constant is unchanged as it is the variance of the random variate (equation (5)).

The Random Variate Times a Constant

We want to determine the mean and variance of the random variate times a constant. Note that if the random variate is the divided by a constant just change the constant to one over the constant. The equation for the first moment of the distribution is...

$$\mathbb{E}\Big[\beta X\Big]=\beta\,\mathbb{E}\Big[X\Big] \tag{10}$$

The equation for the second moment of the distribution is...

$$\begin{aligned}\mathbb{E}\Big[(\beta X)^2\Big] &= \mathbb{E}\Big[\beta^2X^2\Big] \\ &= \beta^2\mathbb{E}\Big[X^2\Big] \end{aligned} \tag{11}$$

The equation for the distribution mean is...

$$\begin{aligned} mean &= \mathbb{E}\Big[\beta X\Big] \\ &= \beta\,\mathbb{E}\Big[X\Big] \end{aligned} \tag{12}$$

The equation for the distribution variance is...

$$
\begin{aligned}
variance &= \mathbb{E}\Big[(\beta X)^2\Big] - \Big[\mathbb{E}\Big[\beta X\Big]\Big]^2 \\
&= \beta^2 \mathbb{E}\Big[X^2\Big] - \Big[\beta\,\mathbb{E}\Big[X\Big]\Big]^2 \\
&= \beta^2 \mathbb{E}\Big[X^2\Big] - \beta^2\Big[\mathbb{E}\Big[X\Big]\Big]^2 \\
&= \beta^2 \Big\{\mathbb{E}\Big[X^2\Big] - \Big[\mathbb{E}\Big[X\Big]\Big]^2\Big\} \qquad (13)
\end{aligned}
$$

Note that the mean of the random variate times a constant is the mean of the random variate (equation (4)) times the constant and the variance of the random variate times a constant is the variance of the random variate (equation (5)) times the square of the constant.

A Constant Times a Random Variate Plus a Constant

We now want to calculate the mean and variance of a constant times a random variate plus another constant. The equation for the first moment of the distribution is...

$$
\begin{aligned}
\mathbb{E}\Big[\beta X + \alpha\Big] &= \mathbb{E}\Big[\beta X\Big] + \mathbb{E}\Big[\alpha\Big] \\
&= \beta\,\mathbb{E}\Big[X\Big] + \alpha \qquad (14)
\end{aligned}
$$

The equation for the second moment of the distribution is...

$$
\begin{aligned}
\mathbb{E}\Big[(\beta X + \alpha)^2\Big] &= \mathbb{E}\Big[\beta^2 X^2 + 2\beta X\alpha + \alpha^2\Big] \\
&= \mathbb{E}\Big[\beta^2 X^2\Big] + \mathbb{E}\Big[2\beta X\alpha\Big] + \mathbb{E}\Big[\alpha^2\Big] \\
&= \beta^2 \mathbb{E}\Big[X^2\Big] + 2\,\beta\alpha\,\mathbb{E}\Big[X\Big] + \alpha^2 \qquad (15)
\end{aligned}
$$

The equation for the distribution mean is...

$$
\begin{aligned}
mean &= \mathbb{E}\Big[\beta X + \alpha\Big] \\
&= \beta\mathbb{E}\Big[X\Big] + \alpha \qquad (16)
\end{aligned}
$$

The equation for the distribution variance is...

$$
\begin{aligned}
variance &= \mathbb{E}\Big[(\beta X + \alpha)^2\Big] - \Big[\mathbb{E}\Big[\beta X + \alpha\Big]\Big]^2 \\
&= \beta^2 \mathbb{E}\Big[X^2\Big] + 2\,\beta\alpha\,\mathbb{E}\Big[X\Big] + \alpha^2 - \Big[\beta\,\mathbb{E}\Big[X\Big] + \alpha\Big]^2 \\
&= \beta^2 \mathbb{E}\Big[X^2\Big] + 2\,\beta\alpha\,\mathbb{E}\Big[X\Big] + \alpha^2 - \beta^2\Big[\mathbb{E}\Big[X\Big]\Big]^2 - 2\,\beta\alpha\,\mathbb{E}\Big[X\Big] - \alpha^2 \\
&= \beta^2 \mathbb{E}\Big[X^2\Big] - \beta^2\Big[\mathbb{E}\Big[X\Big]\Big]^2 \\
&= \beta^2 \Big\{\mathbb{E}\Big[X^2\Big] - \Big[\mathbb{E}\Big[X\Big]\Big]^2\Big\} \qquad (17)
\end{aligned}
$$

Note that the mean of the distribution is the multiplier times the mean of the random variate distribution plus the constant and the variance is the multiplier squared times the variance of the random variate distribution.

A Constant Times The Sum of a Random Variate Plus a Constant

We now want to combine the sections above and calculate the mean and variance of a constant times the sum of the random variate plus another constant. The equation for the first moment of the distribution is...

$$\begin{aligned}\mathbb{E}\Big[\beta(X+\alpha)\Big] &= \mathbb{E}\Big[\beta X + \beta\alpha\Big] \\ &= \mathbb{E}\Big[\beta X\Big] + \mathbb{E}\Big[\beta\alpha\Big] \\ &= \beta\,\mathbb{E}\Big[X\Big] + \beta\,\mathbb{E}\Big[\alpha\Big] \\ &= \beta\,\Big[\mathbb{E}\Big[X\Big] + \alpha\Big] \qquad (18)\end{aligned}$$

The equation for the second moment of the distribution is...

$$\begin{aligned}\mathbb{E}\Big[(\beta(X+\alpha))^2\Big] &= \mathbb{E}\Big[\beta^2 X^2 + 2\beta^2 X\alpha + \beta^2\alpha^2\Big] \\ &= \mathbb{E}\Big[\beta^2 X^2\Big] + \mathbb{E}\Big[2\beta^2 X\alpha\Big] + \mathbb{E}\Big[\beta^2\alpha^2\Big] \\ &= \beta^2\mathbb{E}\Big[X^2\Big] + 2\,\beta^2\alpha\,\mathbb{E}\Big[X\Big] + \beta^2\alpha^2 \qquad (19)\end{aligned}$$

The equation for the distribution mean is...

$$\begin{aligned}mean &= \mathbb{E}\Big[\beta(X+\alpha)\Big] \\ &= \beta\Big[\mathbb{E}\Big[X\Big] + \alpha\Big] \qquad (20)\end{aligned}$$

The equation for the distribution variance is...

$$\begin{aligned}variance &= \mathbb{E}\Big[(\beta(X+\alpha))^2\Big] - \Big[\mathbb{E}\Big[\beta(X+\alpha)\Big]\Big]^2 \\ &= \beta^2\mathbb{E}\Big[X^2\Big] + 2\,\beta^2\alpha\,\mathbb{E}\Big[X\Big] + \beta^2\alpha^2 - \Big[\beta\,\mathbb{E}\Big[X\Big] + \beta\,\alpha\Big]^2 \\ &= \beta^2\mathbb{E}\Big[X^2\Big] + 2\,\beta^2\alpha\,\mathbb{E}\Big[X\Big] + \beta^2\alpha^2 - \beta^2\Big[\mathbb{E}\Big[X\Big]\Big]^2 - 2\,\beta^2\alpha\,\mathbb{E}\Big[X\Big] - \beta^2\alpha^2 \\ &= \beta^2\mathbb{E}\Big[X^2\Big] - \beta^2\Big[\mathbb{E}\Big[X\Big]\Big]^2 \\ &= \beta^2\Big\{\mathbb{E}\Big[X^2\Big] - \Big[\mathbb{E}\Big[X\Big]\Big]^2\Big\} \qquad (21)\end{aligned}$$

Note that the mean of the distribution is the multiplier times the sum of the mean of the random variate distribution plus the constant and the variance is the multiplier squared times the variance of the random variate distribution.

Hypothetical Cases

Case 1:

The distribution of the random variate X is uniform and in the range of zero and one. A uniform distribution over the range zero to one has a mean of 0.5000 a variance of 0.0833. We want to find the mean and variance of 3 times the sum of the uniformly distributed random variate times 5.

The mean of this distribution per equation (20) is...

$$mean = 3 \times (0.5000 + 5) = 16.5 \tag{22}$$

The variance of this distribution per equation (27) is...

$$variance = 3^2 \times 0.08333 = 0.75 \tag{23}$$

Case 2:

A normally-distributed random variate A has a mean of m and a variance of v. We want to convert this random variate into the normally-distributed random variate B with mean zero and variance one. How do we do this?

Mean and Variance of the Random Variate A:

$$mean = \mathbb{E}\Big[A\Big] = m \tag{24}$$

$$variance = \mathbb{E}\Big[A^2\Big] - \Big[\mathbb{E}\Big[A\Big]\Big]^2 = v \tag{25}$$

Proposition: What if we did an affine transformation of A by subtracting the mean of A and dividing by the standard deviation of A. Will that get us to a random variate with mean zero and variance one? To employ the equations above we need definitions for the variables α and β. We will be subtracting the mean so $\alpha = -m$. We will be dividing by the standard deviation so $\beta = 1 \div \sqrt{v}$.

Step 1 - Determine the mean of the new random variate B. We can do this via equation (20) above...

$$\begin{aligned} mean &= \beta\Big[\mathbb{E}\Big[A\Big] + \alpha\Big] \\ &= \frac{1}{\sqrt{v}}\Big[m + (-m)\Big] \\ &= 0 \end{aligned} \tag{26}$$

Step 2 - Determine the variance of the new random variate B. We can do this via equation (27) above...

$$\begin{aligned} variance &= \beta^2\Big[\mathbb{E}\Big[A^2\Big] - \Big[\mathbb{E}\Big[A\Big]\Big]^2\Big] \\ &= \left(\frac{1}{\sqrt{v}}\right)^2\Big[v\Big] \\ &= \frac{1}{v} \times v \\ &= 1 \end{aligned} \tag{27}$$

Conclusion - We can standardize a normally-distributed random variate (i.e. the new random variate has mean zero and variance one) by substracting the mean and dividing by the standard deviation.